CORE-FREE, RANK TWO COSET GEOMETRIES FROM EDGE-TRANSITIVE BIPARTITE GRAPHS

JULIE DE SAEDELEER, DIMITRI LEEMANS, MARK MIXER, AND TOMAŽ PISANSKI

ABSTRACT. It is known that the Levi graph of any rank two coset geometry is an edge-transitive graph, and thus coset geometries can be used to construct many edge transitive graphs. In this paper, we consider the reverse direction. Starting from edge-transitive graphs, we construct all associated core-free, rank two coset geometries. In particular, we focus on 3-valent and 4-valent graphs, and are able to construct coset geometries arising from these graphs. We summarize many properties of these coset geometries in a sequence of tables; in the 4-valent case we restrict to graphs that have relatively small vertex-stabilizers.

1. INTRODUCTION

Most of the time, people working with coset geometries are either analyzing a particular example of a geometry, or taking a group (or a family of groups) with the idea of classifying all geometries, satisfying a set of axioms, where the group has a flag-transitive action [12, 18]. In this paper we use these ideas in the reverse direction. Namely, instead of considering flag-transitive incidence geometries arising from a given group, we use lists of edge-transitive graphs, and study flag-transitive incidence geometries arising from subgroups of their automorphism groups. In particular, we are interested in finding two-transitive geometries. Our computations were made on well-known families of cubic and quartic bipartite graphs. Our approach uses the well-established pathway that was initiated in the 1950's and 1960's by Jacques Tits [35, 36].

This paper addresses multiple fields of mathematics including: group theory, geometry, algebraic graph theory, and combinatorics. For completeness, we recall several known results in each of these fields. The paper is organized as follows. First we give definitions and notation for groups, graphs, and geometries. Subsequently, we provide background results for groups, graphs, and geometries. We also give main observations about flag-transitive incidence geometries and coset geometries. Finally, we provide explanations of our computations, and give some observations regarding the information in our tables, with a link to a web page of data and results.

1.1. **Groups.** We recall some basic notions about groups and give the definitions needed to understand this paper.

The action of a group G on a set X is called *transitive* if there is only one orbit; it is called *primitive* if it is transitive and there is no non-trivial partition of X preserved by

the action. The action is *doubly transitive* if any ordered pair of distinct elements can be mapped pointwise on any other ordered pair by an element of G.

An action is called *faithful* if for any two distinct g, h in G there exists an x in X such that $gx \neq hx$; or equivalently, if for any $g \neq e$ in G there exists an x in X such that $gx \neq x$. Intuitively, different elements of G induce different permutations of X.

We denote the stabilizer of an element p of X by $Stab(G,p)$, which is the subgroup of G that fixes p. A *maximal subgroup* of a group is defined as a proper subgroup such that there is no other proper subgroup containing it.

1.2. **Bipartite graphs and their symmetry.** All graphs in this paper are simple, i.e. without loops or multiple edges. Let X be a graph and G a subgroup of its automorphism group. We consider each edge of X being composed of two opposite *arcs* (sometimes called *darts*, *half-edges*, or *semi-edges*). A *k-arc* is an ordered $(k+1)$-tuple $(\alpha_0, ..., \alpha_k)$ of vertices such that $\{\alpha_{i-1}, \alpha_i\}$ is an edge of X for all $i = 1, ..., k$ and $\alpha_{j-1} \neq \alpha_{j+1}$ for all $j = 1, ..., k-1$. (see [1]); note that a 1-arc is the same as an arc.

We now give the following definitions.

(1) If the group G acts transitively on the set of edges then we call X a *G-edge transitive graph*.
(2) If the group G acts transitively on the set of arcs then we call X a *G-arc transitive graph*.
(3) X is *G-k-arc transitive* if G acts transitively on k-arcs, but not on the $k+1$-arcs (see [1]).
(4) X is *G-half-arc transitive* if G acts transitively on the vertices and on the edges, but not on the arcs (see [5, 27]).
(5) X is *G-semi-symmetric* if G acts transitively on arcs but not transitively on vertices (see [29]).

Note that our definition of a semi-symmetric graph is more general than usual. Namely, we do not require that a semi-symmetric graph be regular. Also note that saying a graph X is G-arc-transitive is the same as saying that X is G-k-arc-transitive for some $k \geq 1$. The notation of G-k-arc-transitive, is the original notation, it also appears noted as (G, k)-arc-transitive.

Let X be a graph with a given vertex coloring. We define three groups of automorphisms of X.

The *full automorphism group*, $Aut(X)$ consists of all automorphisms of X. The group of *color respecting automorphisms*, $Aut_O(X)$, consists of automorphisms of X which preserve the partition of the vertices of X into color classes. The group of *color preserving automorphisms*, $Aut_o(X)$, consists of automorphisms of X such that a vertex and its image are in the same color class.

An automorphism of a bipartite graph X with a given black-and-white coloring that maps black vertices to white vertices is called a *duality*. A duality of order 2 is called *polarity*. If a bipartite graph admits a duality, then it is called *self-dual*; similarly if it admits a polarity, then it is called *self-polar*.

Observe that similar concepts can be defined for edge colored graphs.

However, in this paper we only consider bipartite graphs with a given (black-and-white) vertex coloring.

1.3. **Rank two incidence and coset geometries.** The theory of incidence geometries for general rank has existed for about half a century (see [8, 9, 10, 11, 13, 24, 38]). However, we exclusively consider rank two geometries and adjust the language accordingly.

A rank two *incidence geometry* $\Psi = (P, L, I)$ is a structure consisting of a set P whose elements are called *points*, a set L whose elements are called *lines* and an *incidence relation* I between points and lines. We model such a geometry using a bipartite graph with a given black and white coloring. Black vertices correspond to points and white vertices correspond to lines of the geometry. This graph is called the *incidence graph* or *Levi graph* of the geometry (see [25], pg 5). By interchanging the roles of points and lines in Ψ we obtain the *dual geometry* $\Psi^* = (L, P, I)$. This is equivalent to reversing the colors in the 2-coloring of the Levi graph. The edges of this graph are called *chambers* [36] or *flags* [26] of the geometry. A morphism between two geometries maps points to points, lines to lines, and flags to flags. An isomorphism is a bijective morphism. An isomorphism of Ψ to itself is called an automorphism, and the set of automorphisms form a group, which we denote $Aut_o(\Psi)$. We point out here that this is not the standard notation for incidence geometries, but it allows us to easily go back and forth between automorphisms of graphs and rank two geometries. An isomorphism of Ψ to its dual geometry Ψ^* is called a duality. Finally, an incidence geometry is *connected* if its Levi graph is connected.

A remarkable concept, whose power was discovered by Tits in the 1950's, comes from the requirement that $Aut_o(\Psi)$ acts transitively on the flags of the geometry. We call a geometry Ψ *point-, line-, or flag-transitive* provided that $Aut_o(\Psi)$ acts transitively respectively on points, lines, or flags of the geometry (see [19]). It may happen that the group of automorphisms and dualities acts transitively on flags but the geometry itself is not flag-transitive. In this case we say, that it is *weakly flag-transitive* [28].

Flag-transitive incidence geometries arise from groups and their coset geometries. A rank two *coset geometry* $\Gamma = (G : G_0, G_1)$ is defined by a group G and left cosets of two of its subgroups G_0 and G_1. Given a coset geometry Γ, we define its *underlying incidence geometry* Ψ as follows: points of Ψ are left-cosets of G_0 in G and lines are left-cosets of G_1 in G. A pair of cosets aG_0, bG_1 is incident if and only if $aG_0 \cap bG_1 \neq \emptyset$; also the Levi graph of a coset geometry is defined as the Levi graph of its underlying incidence geometry.

Let $\Gamma = (G : G_0, G_1)$ be a rank two geometry. The group $G_{01} = G_0 \cap G_1$ is called the *Borel subgroup* of the coset geometry $\Gamma = (G : G_0, G_1)$. We call $\mathcal{L}(\Gamma) := \{G_0, G_1, G_{01}\}$ the sublattice (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i \in \{0,1\}}$. The elements of the lattice are called the *parabolic subgroups* and the subgroups G_i's are the *maximal parabolic subgroups.* (see [36]).

Two coset geometries $\Gamma = (G : G_0, G_1)$ and $\Gamma' = (H : H_0, H_1)$ are *isomorphic* if there exists a group isomorphism $f : G \to H$ such that $f(G_0) = H_0$ and $f(G_1) = H_1$. A coset geometry $\Gamma = (G : G_0, G_1)$ is *connected* if its underlying incidence geometry is connected.

We provide an example of a connected rank two coset geometry (see [17]). There exist exactly two geometries $\Gamma(PSL(2, 31) : D_{15}, A_5)$ up to conjugacy, and one up to isomorphism. This geometry is connected since $\langle D_{15}, A_5 \rangle = PSL(2, 31)$. The subgroups D_{15}, A_5 are the maximal parabolic subgroups and the borel subgroup is D_5.

2. Background results

Here we present background results regarding graphs and incidence geometries that are necessary to understand our results. Many of the results are part of the folklore, so proofs will often be omitted.

2.1. **Edge-transitive graphs.** It is well-known (see for instance [1]) that a group G that acts transitively on the set of edges of a graph X has at most two vertex orbits. We state this as a proposition.

Proposition 1. [1] *If G is a group that acts transitively on the set of edges of a connected graph X, then there are at most two orbits on the vertex set. If there are two orbits, then X is bipartite. If there is one orbit, then X is G-vertex-transitive and G acts transitively on the set of arcs of X.*

In this paper we are interested mainly in the case where the action has two orbits: the orbit of black vertices and the orbit of white vertices. If G is too big, i.e. if there is only one orbit, then we select an appropriate subgroup of G, if it exists.

Proposition 2. *Let X be a G-edge-transitive connected graph. Then exactly one of the following is true.*

   (1) *X is G-arc transitive.*
   (2) *X is G-half-arc transitive. (edge- and vertex-transitive but not arc-transitive).*
   (3) *X is G-semi-symmetric. (edge-transitive but not vertex-transitive.)*

Proposition 3. [39] *If X is G-half-arc transitive then it is regular and has an even valence.*

Proposition 4. *Let X be a bipartite graph with a given black and white vertex coloring. Then $Aut_o(X) = Aut_O(X)$ if and only if no automorphism of X maps a black vertex to a white vertex.*

Proof. This follows from the definition of $Aut_o(X)$ and $Aut_O(X)$.   □

Proposition 5. *Let X be a connected bipartite graph with a given black and white coloring. Then $Aut(X) = Aut_O(X)$. A disconnected bipartite graph has $Aut(X) = Aut_O(X)$ if and only if each connected component Y has $Aut_o(Y) = Aut_O(Y)$.*

Hence, for connected bipartite graphs we need to consider only $Aut_o(X)$ and $Aut(X)$.

Proposition 6. *If X is a connected bipartite graph with a given black and white coloring, then either $Aut_o(X) = Aut(X)$ or $Aut_o(X)$ is a subgroup of index 2 in $Aut(X)$ and the graph is self-dual.*

2.2. **Rank two incidence- and coset geometries.** From now on all incidence and coset geometries that we consider are of rank two. Many propositions in what follows rely on this restriction.

Proposition 7. *A rank two incidence geometry that is flag-transitive or weakly flag-transitive is also point- and line-transitive.*

Proof. By definition flag-transitive implies point- and line-transitive. Suppose that the incidence geometry is weakly flag-transitive, in other words $Aut(X)$ is edge-transitive and $Aut_o(X)$ is not edge-transitive. Following [28], weakly flag-transitive is equivalent to half-arc-transitive; which implies point- and line-transitive. □

The following proposition gives a characterization of connected rank two coset geometries.

Proposition 8. *The coset geometry $\Gamma = (G : G_0, G_1)$ is connected if and only if G_0 and G_1 together generate the whole group G.*

Proof. This result is a particular case of Theorem 2 in [19]. □

In general, the number of connected components of a coset geometry is equal to the index of $\langle G_0, G_1 \rangle$ in G. If G_0, G_1 do not generate the whole group G, we may take the group $H = \langle G_0, G_1 \rangle$ and the resulting geometry $\Gamma' = (H : G_0, G_1)$ is a connected component of the original geometry. From now on we assume that we are dealing with connected geometries.

Proposition 9. [37] *Let $\Gamma = (G : G_0, G_1)$ be a coset geometry. The corresponding incidence geometry Ψ is G-flag-transitive.*

3. Characterizations of flag-transitive incidence geometries

In this section, we give two characterizations of flag-transitive incidence geometries. Both of the following propositions are provided by Tits in 1863 (see for example [36]). They can be seen as versions of what is known as the "Tits' Algorithm", which has been developed by Francis Buekenhout (see for example [12, 19] and references).

Proposition 10. *Let $\Psi = (P, L, I)$ be a connected flag-transitive incidence geometry. Then there exists a group $G \leq Aut_o(\Psi)$ and two subgroups G_0 and G_1 with the following three properties.*
  (1) *G acts flag-transitively on Ψ.*
  (2) *G_0 and G_1 together generate G.*
  (3) *The underlying incidence geometry of the coset geometry $\Gamma = (G : G_0, G_1)$ is isomorphic to Ψ.*

The points of Ψ can be labeled by left cosets of G_0 in G, the lines of Ψ can be labeled as left cosets of G_1 in G, and the flags of Ψ can be labeled by left cosets of $G_{01} = G_0 \cap G_1$ in G.

Proposition 11. *Let X be a connected bipartite edge-transitive graph with a given black and white coloring such that $Aut_o(X)$ acts transitively on the edges of X, and let b be a black vertex incident to a white vertex w. For every subgroup H of $Aut_o(X)$ that acts transitively on the edges of X, there exists a coset geometry $\Gamma = (H : H_0, H_1)$ whose underlying incidence geometry has Levi graph isomorphic to X, with $H_0 = H \cap Stab(Aut_o(X), b)$, and $H_1 = H \cap Stab(Aut_o(X), w)$.*

Using the above proposition, we see that from a flag-transitive incidence geometry ψ, we can define many associated coset geometries. One of these coset geometries is the largest core-free coset geometry, which we denote by Γ_Ψ. To constuct these geometry Γ_Ψ,

choose any black vertex b and adjacent white vertex w, $G = Aut_o(X)$, $G_0 = Stab(G, b)$, $G_1 = Stab(G, w)$. Any other black vertex can be chosen, which gives an isomorphic construction.

Now we define a *stable* coset geometry $\Gamma = (H : H_0, H_1)$ to be a coset geometry with the property that $\Gamma_{\Psi(\Gamma)} \cong \Gamma$. In 3.1, we give an example to show that not all coset geometries are stable. We also could define a *stable* edge-transitive incidence geometry to be with the property $\Psi(\Gamma_\Psi) \cong \Psi$, but we notice that all edge-transitive incidence geometries are stable. In other words, by combining the two above propositions, we observe that if we start with a connected flag-transitive incidence geometry, and we construct an associated coset geometry, the incidence geometry of this coset geometry is the same as the original one.

Each coset geometry Γ defines a unique incidence geometry $\Psi(\Gamma)$, but in general, the converse is false. This leads us to the following proposition in the connected case.

Proposition 12. *If Ψ is a connected rank two incidence geometry, then there is either no coset geometry that has Ψ as its underlying incidence geometry, or there are infinitely many non-isomorphic coset geometries that have Ψ as their underlying incidence geometry.*

Proof. Let Ψ be a connected rank two incidence geometry. By Proposition 9, if Ψ is not flag transitive, then there is no coset geometry that has Ψ as its underlying incidence geometry. Assume that Ψ is flag transitive. By Proposition 10, there is a coset geometry $\Gamma_\Psi = (G : G_0, G_1)$ with underlying incidence geometry isomorphic to Ψ. Now let $\Gamma' = (G \times H : G_0 \times H, G_1 \times H)$, which is not isomorphic to Γ as a coset geometry (as long as H is nontrivial). However, they yield the same incidence geometry Ψ. $\square$

Proposition 13. *Let $\Gamma = (G : G_0, G_1)$ be a connected coset geometry and Ψ its incidence geometry. The G-stabilizer of any point of Ψ is isomorphic to G_0 and the G-stabilizer of any line of Ψ is isomorphic to G_1. The stabilizer of each flag (fixing both endpoints) is isomorphic to $G_{01} = G_0 \cap G_1$.*

Proof. Following Proposition 9, the incidence geometry Ψ is G-flag-transitive, which by Proposition 11 implies that the G-stabilizer of any point of Ψ is isomorphic to G_0 and also that the G-stabilizer of any line of Ψ is isomorphic to G_1.

Let $G_{01} = Stab(Aut_o(X), \{p, l\})$, be the stabilizer of a flag. Assume now that $G'_{01} = G_0 \cap G_1$. If $\alpha \in G'_{01}$ then $\alpha(p) = p$ and $\alpha(l) = l$ which implies that $\alpha(\{p, l\}) = \{p, l\}$. Hence $G'_{01} \leq G_{01}$. On the other side, if $\alpha \in G_{01}$ then $\alpha(\{p, l\}) = \{p, l\}$. Since the automorphisms are color preserving it implies that $\alpha(p) = p$ and $\alpha(l) = l$. Hence $G_{01} \leq G'_{01}$. $\square$

The proof of this proposition implies that if we require that $G \cong G'$, then two coset geometries $\Gamma = (G : G_0, G_1)$ and $\Gamma' = (G' : G'_0, G'_1)$, with the same incidence geometries, are always isomorphic.

Given the Levi graph X of a coset geometry $\Gamma = (G : G_0, G_1)$, the group G is not always isomorphic to a subgroup of $Aut(X)$. There is still a natural action of G on X; however this action is not always faithful. The following theorem gives a necessary and sufficient condition for a faithful action.

Theorem 1. *Let $\Gamma = (G : G_0, G_1)$ be a connected coset geometry with Levi graph X; then $G_0 \cap G_1$ is core-free in G if and only if G acts faithfully on the edge set of X.*

Proof. In [20] it is proved that if $G_0 \cap G_1$ is core-free in G, then G acts faithfully on the cosets of G_0 and the cosets of G_1 in G, and thus acts faithfully on the edges of X. To prove the converse, let $G_{01} = G_0 \cap G_1$ and assume G_{01} is not core-free in G. This implies there exists a nontrivial element $g \in \bigcap_{f \in G} f G_{01} f^{-1}$. Thus, this g fixes left cosets fG_{01} for all $f \in G$. As we saw in Proposition 10, these left cosets determine the edges of X. Therefore g fixes all the edges of X, and the action of G on X is not faithful. $\square$

We denote any rank two coset geometry $\Gamma = (G : G_0, G_1)$ with the property that $G_0 \cap G_1$ is core-free in G as a *core-free coset geometry.*

Corollary 1. [20] *Let X be the Levi graph of a core-free coset geometry $\Gamma = (G : G_0, G_1)$. Then G is a subgroup of $Aut_0(X)$, and X is G-edge transitive and G-vertex intransitive.*

3.1. **A non-stable coset geometry.** As we have noted, incidence geometries are more general objects than coset geometries. Moreover, if an incidence geometry is flag-transitive it gives rise to a coset geometry. However, it can happen that many non-isomorphic coset geometries have the same associated incidence geometry. We have already seen this to be the case where G is too large to be isomorphic to a subgroup of $Aut_o(\Psi)$. It can also happen, if Ψ is the associated incidence geometry for a coset geometry $\Gamma = (G : G_0, G_1)$, that G is a proper subgroup of $Aut_o(\Psi)$, and thus the coset geometry constructed using Proposition 11 is not isomorphic to Γ.

To demonstrate this, we consider the 1-skeleton of the hemidodecahedron as a coset geometry (see for example [7]). More precisely, let $G \cong Alt(5)$ be generated by $s_0 = (2,3)(4,5)$, $s_1 = (1,2)(3,4)$, and $s_2 = (2,5)(3,4)$. If we let $G_0 = \langle s_1, s_2 \rangle \cong S_3$ and $G_1 = \langle s_0, s_2 \rangle \cong 2^2$, then the incidence graph for the coset geometry $\Gamma = (Alt(5) : G_0, G_1)$ is isomorphic to the subdivision of the Petersen graph. Thus, $Aut_o(\Psi) \cong Sym(5)$, and we can construct a new coset geometry $\Gamma' = (Sym(5), H_0, H_1)$ as demonstrated in Proposition 11.

3.2. **Isomorphism and Conjugacy.** We can choose different notions of what it means for geometries to be equivalent. We have defined isomorphisms of incidence geometries as a color-preserving isomorphisms of bipartite graphs. For two coset geometries $\Gamma = (G : G_0, G_1)$ and $\Gamma' = (G' : G_0', G_1')$ to be equivalent, it must be the case that the groups $G \cong G'$.

Then, we have at least one new way of defining equivalence between coset geometries. We recall that two coset geometries $\Gamma = (G : G_0, G_1)$ and $\Gamma' = (G : G_0', G_1')$ are defined to be isomorphic if there is an element of the automorphism group of G that sends G_0 and G_1 to G_0' and G_1' respectively. On the other hand, we define two coset geometries Γ and Γ' to be conjugate if there is an element of the inner automorphism group of G that sends G_0 and G_1 to G_0' and G_1' respectively.

Proposition 14. *There exits isomorphic coset geometries which are not conjugate.*

Proof. We prove this proposition by providing the following example (see [6]). Let $\Gamma = (M_{12} : K(99), K(95))$ and $\Gamma' = (M_{12} : \overline{K(99)}, h^{-1}K(95)h)$; here $K(99)$ is isomorphic

to $M_9 \ltimes S_3$, $\overline{K(99)}$ is isomorphic to a non-conjugate copy of $M_9 \ltimes S_3$, $K(95)$ is isomorphic to $M_8 \ltimes S_4$, and h is any element of M_{12}. In this example Γ and Γ' are isomorphic, but not conjugate. Although G_1 and G'_1 are conjugate in G, G_0 and G'_0 are not.   □

4. Geometric properties of edge-transitive bipartite graphs

For a given bipartite, edge-transitive graph X with a fixed black and white coloring, we may try to determine all subgroups H of $Aut_o(X)$ that act transitively on the edge set of X. As we have seen earlier, there are three more groups of interest here: H_0, H_1, H_{01}: the stabilizer in H of a black vertex, of a white vertex, and of an edge, respectively.

There are many axioms to verify on coset geometries (see for example [12]), therefore it is important to investigate the relations between the groups H_0, H_1, H_{01} and H for each coset geometry. Indeed, for each transitive subgroup H we ask the following:

(1) Is H_{01} maximal in H_0? Does H_0 act doubly transitively on the set of neighbors of a black vertex?
(2) Is H_{01} maximal in H_1? Does H_1 act doubly transitively on the set of neighbors of a white vertex?
(3) Is H_0 maximal in H? Does H act doubly transitively on the set of black vertices?
(4) Is H_1 maximal in H? Does H act doubly transitively on the set of white vertices?
(5) Is H normal in $Aut_o(X)$?
(6) Is H_0 normal in $Aut_o(X)$?
(7) Is H_1 normal in $Aut_o(X)$?
(8) Is H_{01} normal in $Aut_o(X)$?
(9) Is X H-self-dual? If so, what is the smallest order of a duality?
(10) What is the maximum order of an element of H?

The answer to these questions for many graphs can be found online at:

http://dev.ulb.ac.be/solvay/math/geomgr.html

5. Lists of edge-transitive graphs

The most well-known census of arc-transitive trivalent graph is the Foster Census [4]. It contains arc-transitive cubic graphs and has been recently vastly extended by Marston Conder [16]. The case of semi-symmetric cubic graph was studied in [15]. Iofinova and Ivanov [23] showed that there exist exactly five bipartite cubic semisymmetric graphs whose automorphism groups preserves the bipartite parts and acts primitively on each part. For each bipartite graph X in these censuses, we have constructed all possible core-free coset geometries with Levi graph isomorphic to X.

Quartic edge-transitive graphs have been extensively studied by Potočnik in [30], and a census of these graphs can be found in [33]. For each bipartite graph X in this census which has a vertex stabilizer of size less than 100, we have constructed all possible core-free coset geometries with Levi graph isomorphic to X.

6. Rank two geometries and configurations of points and lines

Rank two geometries that have semi-regular bipartite Levi graphs of girth at least 6 are also known as combinatorial configurations of points and lines. For such geometries a natural question to ask is whether they admit geometric realization as configurations

of points and lines in the Euclidean plane. Edge-transitive bipartite Levi graphs give some promises that methods of polycyclic configurations [2] could apply in many cases. The case of quartic half-arc-transitive Levi graphs has been studied in [28]. For basics of geometric configurations of points and lines the reader is referred to Grünbaum [21].

An interesting question arises from the study of cyclic configurations; see, for instance, [21, 22].

Problem 1. *Determine all cubic and quartic flag-transitive cyclic configurations.*

7. EXPERIMENTAL RESULT

Using Magma [3], for each graph X in various families of edge transitive graphs, a number of coset geometries are constructed using edge transitive subgroups H of $Aut_o(X)$. The groups H_0, H_1, and H_{01} are constructed as in Proposition 11. For each coset geometry we are interested in the relationships between the groups. Some of this information is contained in the columns labeled "Max" and "Norm."

The "Max" column has a sequence of four elements, corresponding to the four group relationships $H_{01} \leq H_0$, $H_{01} \leq H_1$, $H_0 \leq H$, and $H_1 \leq H$ respectively. The "Max" column has a sequence of four elements from the possibilities: "M" which stands for Maximal but not two transitive, "2T" which stands for 2-transitive, and "X" which stands for neither maximal nor 2-transitive. The "Norm" column has a sequence of four elements, corresponding to the four group relationships $H \leq Aut_o(X)$, $H_0 \leq Aut_o(X)$, $H_1 \leq Aut_o(X)$, and $H_{01} \leq Aut_o(X)$ respectively. The "Norm" column tells which of these four subgroups are normal. The "Dual" column shows if the coset geometry is self-dual, and if so the minimal order of a duality. Finally, the "MaxOrd" column tells the maximum order of an element in H.

Here, we provide further analysis of the "Max" and "Norm" columns (for both 3- and 4-valent symmetric graphs), giving the number of times each outcome appears, and the structure of groups in unique cases. We also give two explanations of why some possibilities in this column do not appear.

3-valent Sym.	Max
2T2TXX	315
2T2T2T2T	4
MMXX	587
M2TXX	32
2T2TMM	21
M2T2TM	1
2TMXX	32
2TMM2T	1
MMMM	88
MM2T2T	1

3-valent Sym.	Norm
YNYN	1
YYYY	1
YNNY	674
NNNY	1
YYNN	1
YNNN	404

4-valent Sym.	Max
2TXX2T	11
2TXXM	1
X2TXX	26
2TXMM	1
2TXXX	19
XX2T2T	1
XXMM	17
2T2TMM	5
X2T2TX	11
2T2TXX	38
XXXX	1503
2T2T2T2T	10

4-valent Sym.	Norm
YNNN	466
NNYN	4
YNNY	525
NYNN	4
NNYY	3
NYNY	3
YYNN	2
YYYY	1
YNYN	2
NNNN	365
NNNY	268

In the 3-valent symmetric case, if we look at all possible sequences of four elements in the "Max" column, we can rule out the sequences with an X in the first or second position. Indeed, these are 3-valent graphs, which implies that H_{01} is always maximal in H_0 and in H_1, since 3 is prime.

In the census of 3-valent symmetric graphs, three of the ten possible sequences in the "Max" column only appear once. Two of them ("2TMM2T" and "M2T2TM")

arise from coset geometries that are duals of each other: $(S_3 \times C_3 : C_3 \times 2, S_3)$ and $(S_3 \times C_3 : S_3, C_3 \times 2)$, with $H_{01} = C_2$. These geometries arise from the complete bipartite graph $K_{3,3}$. The third possibility "MM2T2T", comes from the cubical graph, giving the unique geometry $(A_4 : C_3, C_3)$. We note that none of these three sequences appears in the 4-valent case.

Looking at the 1643 geometries that we analyze from 4-valent symmetric graphs, we make the observation that M cannot appear as one of the first 2 elements in the "Max" column. This follows from the fact that the number of cosets of H_{01} in H_i with $i = 0, 1$ equals four; also if H_{01} is a maximal subgroup of H_i with $i = 0$ or 1, then H_i acts primitively on the cosets of H_{01}. There are only two primitive groups acting on 4 points, namely A_4 and S_4, both of which are 2-transitive groups.

There are three sequences that appear only once in the "Max" column. The sequence "XX2T2T" arises from the graph C4[10.2] and gives the coset geometry $((C_5 : C_4) : C_4, C_4)$.

The two others arise from the graph C4[110.7] giving the coset geometries $(PGL(2, 11) : S_4, D_6)$ and $(PSL(2, 11) : A_4, D_3)$.

When "YYYY" appears in the "Norm" column there is a unique geometry in our list coming from a 4-valent graph, and a unique geometry in our list coming from a 3-valent graph. The graph "C4[8.1]" in our tables is the complete bipartite graph $K_{4,4}$. It gives rise to the coset geometry $(C_2^2 \times C_2^2 : C_2^2, C_2^2)$.

Similarly, the complete bipartite graph $K_{3,3}$ gives rise to a few coset geometries that are unique in the "Norm" column $(C_3 \times C_3 : C_3, C_3)$ for "YYYY", $(S_3 \times C_3 : S_3, C_3 \times 2)$ for "YYNN", and $(S_3 \times C_3 : C_3 \times 2, S_3)$ for "YNYN".

When "NNNY" appears in the "Norm" column there is a unique geometry in our list coming from a 3-valent graph. The Heawood graph C14.1 provides the unique coset geometry in our table $((C_7 : C_3) : C_3, C_3)$.

We also point out that the "Max" column does not fix the "Norm" column (and vise versa). This can be seen through the previous examples.

8. CONCLUDING REMARKS

In this paper we established an exact connection between rank two core-free coset geometries and two-colored edge-transitive graphs. We focused on example of 3 and 4 regular graphs. When constructing coset geometries from 4-valent edge transitive graphs, we noticed that some of these graphs have abundantly large automorphism groups. More precisely, the stabilizer of a vertex is very large in these cases. Many of these graphs are labeled as "unworthy graphs" in [33]; however there are other graphs with very large automorphism groups that are "worthy." These 4-valent graphs, with large automorphism groups, have been studied in [34]. Also, in [31, 32] it is shown that if a graph with such a large automorphism group is also arc transitive, then it belongs to a well understood infinite family of graphs.

This leads to an interesting question.

Problem 2. *Classify the semi-symmetric graphs with large vertex stabilizers. Similarly, classify core-free rank two coset geometries coming from these graphs.*

9. ACKNOWLEDGEMENT

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Université Libre de Bruxelles Département de Mathématiques - C.P.216, Boulevard du Triomphe, B-1050 Bruxelles
E-mail address, J. De Saedeleer: judesaed@ulb.ac.be

Université Libre de Bruxelles Département de Mathématiques - C.P.216, Boulevard du Triomphe, B-1050 Bruxelles

Université Libre de Bruxelles Département de Mathématiques - C.P.216, Boulevard du Triomphe, B-1050 Bruxelles

Slovenija University of Ljubljana, Inštitut za matematiko, fiziko in mehaniko Jadranska 19, Ljubljana